ISRAEL JOURNAL OF MATHEMATICS **128** (2002), 355–379

ELEMENTARY THEORIES OF COMPLETELY SIMPLE SEMIGROUPS

BY

BELLA V. ROZENBLAT

Department of Mathematics, Department of Computer Science
Ben-Gurion University of the Negev, POB 653, Beer-Sheva, 84105, Israel
e-mail: bella@cs.bgu.ac.il

ABSTRACT

The connections between first-order formulas over a completely simple semigroup C and corresponding formulas over its structure group H are found in this paper. For the case of finite sandwich-matrix the criterion of decidability of the elementary theory $T(C)$ is established in terms of the elementary theory of H in the enriched signature (Theorem 1). For the general case the criterion is established in terms of two-sorted algebraic systems (Theorem 2). Sufficient conditions in terms of H for decidability and for undecidability of $T(C)$ are outlined. Corollaries and examples are presented, among them an example of a completely simple semigroup with a finite structure group and with undecidable elementary theory (Theorem 3).

Introduction

In the sea of today's research on semigroups the current of completely simple semigroups occupies quite an important place. A comprehensive survey on the subject, including brand new results, is given in [P-R]. The present paper is devoted to the problem of decidability for elementary theories of completely simple semigroups. The first results obtained in this direction concerned relatively free completely simple semigroups, and were announced in [R92] and [R93].

In the present work we investigate the elementary theory of an arbitrary completely simple semigroup $M(H, I, J, P)$ and its connection with the theory of the corresponding structure group H. It is easy to translate every closed first-order

Received December 20, 1999 and in revised form February 3, 2001

formula φ of the group signature into a corresponding closed first-order formula φ^* of the semigroup signature in such a way that φ is valid on a group H iff φ^* is valid on a Rees matrix semigroup $C = M(H, I, J, P)$. Such a translation is called **the exact interpretation** of H in C and, in the case of undecidable $T(H)$, it implies the undecidability of $T(C)$. The original aim of this research was to look for ways to translate in the opposite direction, i.e., to try to pass from an arbitrary first-order formula on C to an equivalent formula on H, for several natural signatures of C and H. The reason is that if for some fixed signatures of C and H there exist such translations in both the forward and backward directions, then for these signatures the algorithmic behavior of the elementary theories of C and H is similar, i.e., the two theories are both decidable or both undecidable in this case. The present paper is the implementation of the original intentions under certain conditions.

More precisely, in Section 2 we exactly interpret H in C (in Proposition 1, for the signature $< \cdot >$, in Proposition 3, for some other signatures and for a normalized sandwich-matrix) and, for finite I and J, we exactly interpret C in $H \times I \times J$ for a convenient signature (Proposition 2). That enables us to find sufficient conditions and necessary conditions for the decidability of $T(C)$, and, in the case of finite I and J and a normalized sandwich-matrix, to establish the following criterion: this decidability is equivalent to the decidability of $T(H)$ with constant symbols for elements belonging to P, and also it is equivalent to the decidability of $T(H)$ with the unary predicate $p(x)$ for membership in P (Theorem 1).

The results of Section 3 are obtained for the general case, when I and J are not necessarily finite. We introduce for $i_0 \in I$, $j_0 \in J$ the notion of the two-sorted algebraic system $D(i_0, j_0)$. It has basic sets H and $I \times J$ with the corresponding multiplication operation inside each of these sets and with the function π: $I \times J \to H$, which corresponds to the sandwich-matrix, obtained from P by the normalization according to the j_0-th row and i_0-th column. Then we exactly interpret C in $D(i_0, j_0)$ (Lemma 3) and we interpret $D(i_0, j_0)$ in C with the corresponding constant in the signature (Lemma 4). That enables us to establish the criterion of decidability of $T(C)$ in the general case: this decidability is equivalent to the decidability of the elementary theory of the class of all systems $D(i_0, j_0)$ (Theorem 2).

In Section 4 we give an example of a completely simple semigroup C_0 over a three-element group such that $T(C_0)$ is undecidable (Theorem 3). This theorem is based on Theorem 2, and shows that the finiteness of I and J is a vital condition for Theorem 1.

In Section 1 we formulate propositions and theorems and then prove them in Sections 2, 3 and 4. We also deduce corollaries, among them:

– the undecidability of the elementary theory for a free non-monogenic completely simple semigroup

1) of the variety of all completely simple semigroups over metabelian groups,

2) of the variety $\mathcal{CS}(\mathcal{N}_m)$ of all completely simple semigroups over groups from $\mathcal{N}_m$, where $\mathcal{N}_m$ is the variety of all nilpotent groups of class $m > 1$ (Corollary 1.1);

– the decidability of the elementary theory for a free finitely generated completely simple semigroup

1) of any variety of completely simple semigroups over abelian groups,

2) of the variety $\mathcal{CS}(\mathcal{N}_m^{q^k})$ of all completely simple semigroups over groups from $\mathcal{N}_m^{q^k}$ with q prime and $m < q$, where $\mathcal{N}_m^{q^k}$ is the variety of all nilpotent groups of class m and of exponent q^k (Corollary 1.3);

– the decidability of the elementary theory for any completely simple semigroup with a finite number of maximal subgroups, which are abelian (Corollary 1.2), and other corollaries.

ACKNOWLEDGEMENT: I am grateful to my teacher L. Shevrin; in particular, it was his question in [Sv] about semigroup varieties with decidable elementary theories of relatively free semigroups that lead me finally to investigate elementary theories of arbitrary completely simple semigroups. I thank N. Reilly, whose interest in my talk at the Workshop on Semigroups, Formal Languages and Combinatorics on Words (Kyoto, 1992) gave additional inspiration to my research, and U. Abraham, R. Lipyansky, G. Mashevitzky and S. Smith, whose advice helped me to improve the style of the paper. Thanks are also due to the referee, whose deep comments and questions stimulated me to clarify proofs, to add important remarks, to correct a lot of typos, and to think about a possible generalization of the techniques of this paper to other classes of semigroups in the future.

1. Definitions, notation, formulation of main results and corollaries

Throughout this paper we use the following notation:

$\rightleftharpoons$ — "equals by definition";

L_σ — the class of all first-order formulas of a signature σ (the signature may contain constant symbols, symbols for operations and for predicates);

$T_\sigma(A)$ — the elementary theory of the algebraic system A of the signature σ (this means the class of all closed formulas from L_σ that are valid on A);

$A \models \varphi$ — the formula φ is valid on A.

In the terminology concerning first-order formulas and theories we follow [Ma]; in particular, the term **algebraic system** is equivalent to **algebraic structure**, and the term **decidability** is equivalent for $T_\sigma(A)$ to **solvability** or **recursivity** and means the existence of an algorithm answering, for any closed formula $\varphi \in L_\sigma$, whether $\varphi \in T_\sigma(A)$ or not.

We will use the following definitions, notations and facts concerning completely simple semigroups [C-P].

Let C be a completely simple semigroup, and let $C \simeq M(H, I, J, P)$ be a representation of C by a Rees matrix semigroup, where H is the structure group of C, I and J are index sets, and $P \rightleftharpoons (p_{ji} \mid j \in J, i \in I)$ is the sandwich-matrix. Then C is isomorphic to the set of all triples $\{(h, i, j) \mid h \in H, i \in I, j \in J\}$ with binary operation $\cdot$, defined by

$$(h_1, i_1, j_1) \cdot (h_2, i_2, j_2) = (h_1 \cdot p_{j_1 i_2} \cdot h_2, i_1, j_2),$$

where $h_1 \cdot p_{j_1 i_2} \cdot h_2$ denotes the usual product in the group H.

In all our proofs we will identify the elements of C with such triples for a fixed representation $M(H, I, J, P)$. Our main results, however, do not depend on a particular Rees representation of a completely simple semigroup C (see Remarks 1 and 3).

Let e be the identity element of H. The sandwich-matrix P is called **normalized** if for some $i_0 \in I, j_0 \in J$ all the elements of the i_0-th column and all the elements of the j_0-th row of P equal e. In this case we will say also that P is (i_0, j_0)-**normalized.**

For every $i_0 \in I, j_0 \in J$ let

$$P(i_0, j_0) \rightleftharpoons ((p_{ji_0})^{-1} \cdot p_{ji} \cdot (p_{j_0 i})^{-1} \cdot p_{j_0 i_0} \mid j \in J, i \in I).$$

Then

$$C \simeq M(H, I, J, P) \simeq M(H, I, J, P(i_0, j_0)),$$

and $P(i_0, j_0)$ is (i_0, j_0)-normalized.

Let $E \rightleftharpoons \{((p_{ji})^{-1}, i, j) \mid i \in I, j \in J\}$; for any $i_0 \in I, j_0 \in J$ let

$$H(i_0, j_0) \rightleftharpoons \{(h, i_0, j_0) \mid h \in H\}, \quad e(i_0, j_0) \rightleftharpoons ((p_{j_0 i_0})^{-1}, i_0, j_0).$$

Then E is the set of all idempotents of C, $H(i_0, j_0)$ is the maximal subgroup of C containing the identity element $e(i_0, j_0)$, and the mapping

$$h \mapsto (h \cdot (p_{j_0 i_0})^{-1}, i_0, j_0)$$

defines an isomorphism of H onto $H(i_0, j_0)$ [C-P].

In the paper we usually formulate and prove the statements for the simplest possible signatures of H and C; all the results remain true after adding the unary operation symbol ($^{-1}$) and the constant symbol for the identity element (e) to the signature of H and (or) symbols for two unary operations ($^{-1}$) and (^{0}) to the signature of C (see the following remark).

Remark 0: The expression $(y = x^{-1})$ is equivalent on H to $(x \cdot y = e)$; the expression $(x = e)$ is equivalent on H to $(x \cdot x = x)$. Therefore, every elementary formula on H that contains ($^{-1}$) and/or e can be effectively transformed to an equivalent elementary formula without ($^{-1}$) and e. Hence if a signature σ_1 of H contains the multiplication symbol and $\sigma_1^* \rightleftharpoons \sigma_1 \cup \{(^{-1}), e\}$ is the signature obtained by adding the symbols ($^{-1}$) and e to σ_1, then $T_{\sigma_1^*}(H)$ is decidable iff $T_{\sigma_1}(H)$ is decidable. For $x \in C$ let x^0 denote the identity element of the maximal subgroup of C that contains x, and let x^{-1} denote the inverse to x in this subgroup. The expression $(y = x^{-1})$ is equivalent on C to $(x \cdot y = y \cdot x = x^0)$; the expression $(y = x^0)$ is equivalent on C to $(x \cdot y = y \cdot x = x)$. Therefore, every elementary formula on C that contains ($^{-1}$) and/or (^{0}) can be effectively transformed to an equivalent elementary formula without ($^{-1}$) and (^{0}). Hence if a signature σ_2 of C contains the multiplication symbol and $\sigma_2^{**} \rightleftharpoons \sigma_2 \cup \{(^{-1}), (^{0})\}$ is the signature obtained by adding the symbols ($^{-1}$) and (^{0}) to σ_2, then $T_{\sigma_2^{**}}(C)$ is decidable iff $T_{\sigma_2}(C)$ is decidable.

Note that varieties of completely simple semigroups are usually considered as defined by identities of the signature $< \cdot, ^{-1} >$; we mean that in the present paper too.

In this section we formulate the theorems and most of the propositions of the paper that concern the questions of decidability for elementary theories (their proofs are given in Sections 2, 3 and 4). We also discuss here the results obtained and deduce corollaries.

Let us begin with a simple but important result.

PROPOSITION 1: *If $T_{<\cdot>}(H)$ is undecidable, then $T_{<\cdot>}(C)$ is undecidable.*

This means that decidability of $T_{<\cdot>}(H)$ is necessary for decidability of $T_{<\cdot>}(C)$. But it is not sufficient, as will be shown below in Corollary 1.5 of Theorem 1 and in the Example given in Section 4.

COROLLARY 1.1: *Let C be a free non-monogenic completely simple semigroup of a variety $\mathcal{V}$ satisfying one of the following two conditions:*

1. *$\mathcal{V}$ is the variety of all completely simple semigroups over metabelian groups;*

2. *$\mathcal{V}$ is the variety $\mathcal{CS}(\mathcal{N}_m)$ of all completely simple semigroups over groups from $\mathcal{N}_m$, where $\mathcal{N}_m$ is the variety of all nilpotent groups of class $m > 1$.*

Then $T_{<\cdot>}(C)$ is undecidable.

Proof: If $\mathcal{V}$ satisfies (1) or (2), then, according to [J], the structure group H of C is a free metabelian (or a free nilpotent of class m, respectively) group with more than one generator. In both cases $T_{<\cdot,^{-1},e>}(H)$ is undecidable: for the case (1) it follows from the results of [Ro], in the case (2) it was proved in [D]. Hence $T_{<\cdot>}(H)$ is undecidable too (see Remark 0), and $T_{<\cdot>}(C)$ is undecidable by Proposition 1. ∎

The following Proposition gives a sufficient condition for decidability of $T_{<\cdot>}(C)$ in the case of finite sandwich-matrix P.

PROPOSITION 2: *Let I and J be finite sets. If $T_{<\cdot,a_1,\dots,a_n>}(H)$ is decidable, where $a_1, \dots, a_n$ are symbols for all distinct elements of H belonging to P, then $T_{<\cdot>}(C)$ is decidable.*

In the notation of Proposition 2, let σ_P be $<\cdot, a_1, \dots, a_n>$. Corollary 1.4 below shows for finite I and J that the decidability of $T_{\sigma_P}(H)$ is not necessary for decidability of $T_{<\cdot>}(C)$. But if P is normalized, then the two theories are both decidable or both undecidable (Theorem 1).

COROLLARY 1.2:

1. *If I and J are finite sets and H has a decidable elementary theory in the group signature with constant symbols for all elements of H, then for every $J \times I$ matrix P over H the elementary theory of $M(H, I, J, P)$ in the semigroup signature $< \cdot >$ is decidable.*
2. *If a completely simple semigroup C has a finite number of maximal subgroups and they are abelian, then $T_{<\cdot>}(C)$ is decidable.*

Proof: Statement (1) follows directly from Proposition 2. Statement (2) follows by Proposition 2 from the decidability of $T_{<\cdot,a_1,\dots,a_n>}(H)$ for any abelian group H and elements $a_1, \dots, a_n$ [M]. ∎

COROLLARY 1.3: *Let C be a free finitely generated completely simple semigroup of a variety $\mathcal{V}$ satisfying one of the following two conditions:*

1. *$\mathcal{V}$ is a variety of completely simple semigroups over abelian groups;*
2. *$\mathcal{V}$ is the variety $\mathcal{CS}(\mathcal{N}_m^{q^k})$ of all completely simple semigroups over groups from $\mathcal{N}_m^{q^k}$ with q prime and $m < q$, where $\mathcal{N}_m^{q^k}$ is the variety of all nilpotent groups of class m and of exponent q^k.*

Then $T_{<\cdot>}(C)$ is decidable.

Proof: If $\mathcal{V}$ satisfies (1) or (2), then, according to [J] and [Ra],

$$C \simeq M(H, I, J, P),$$

where H is some abelian group (or, a free finitely generated group from $\mathcal{N}_m^{q^k}$, respectively), and I and J are finite sets. In case (1), $T_{<\cdot>}(C)$ is decidable by Corollary 1.2 (2). In case (2), the decidability of $T_{<\cdot>}(C)$ follows by Proposition 2 from the decidability of the elementary theory with finitely many constants of the group H, proved *de facto* in [B]. ∎

In the next corollary we will use a group H having elements $a_1, a_2, \ldots, a_n$, such that the following condition holds:

(*) $T_{<\cdot>}(H)$ is decidable, but $T_{<\cdot, a_1, \ldots, a_n>}(H)$ is undecidable.

(Concerning examples of such groups H, see Theorems 3 and 10 from [M-R]; simpler examples may be constructed too.)

Corollary 1.4: *Let a group H and elements $a_1, a_2, \ldots, a_n \in H$ satisfy (*), and let*

$$I \rightleftharpoons \{1, 2, \ldots, n\}, \quad J \rightleftharpoons \{1\}, \quad P \rightleftharpoons (a_1\ a_2\ \ldots\ a_n),$$
$$C \rightleftharpoons M(H, I, J, P).$$

Then $T_{<\cdot>}(C)$ is decidable, although $T_{<\cdot, a_1, \ldots, a_n>}(H)$ is undecidable.

Proof: $C \simeq M(H, I, J, P')$, where $P' \rightleftharpoons (e\ e\ \ldots\ e)$ is the normalized form of P (see the definition of $P(i_0, j_0)$ at the beginning of the section). From the decidability of $T_{<\cdot>}(H)$ it follows that $T_{<\cdot, e>}(H)$ is decidable (by Remark 0). The decidability of $T_{<\cdot, e>}(H)$ implies by Proposition 2 the decidability of $T_{<\cdot>}(M(H, I, J, P'))$, i.e., of $T_{<\cdot>}(C)$. ∎

This corollary shows that Proposition 2 cannot be reversed. It also gives an example of two different Rees matrix representations $M(H, I, J, P)$ and $M(H, I, J, P')$ of the same completely simple semigroup, where I and J are finite, $T_{\sigma_P}(H)$ is undecidable and $T_{\sigma_{P'}}(H)$ is decidable. Such a situation is possible only if P is not normalized (see Remark 1 below).

The following Theorem 1 gives the criterion for decidability of $T_{<\cdot>}(C)$ in the case when the sandwich-matrix P is finite and normalized.

THEOREM 1: *Let I and J be finite sets and P be a normalized $J \times I$ matrix over a group H. Then for $C \simeq M(H,I,J,P)$, the following three conditions are equivalent:*

1. *$T_{<\cdot>}(C)$ is decidable.*
2. *$T_{<\cdot,a_1,\ldots,a_n>}(H)$ is decidable, where $a_1,\ldots,a_n$ are symbols for all distinct elements of H belonging to P.*
3. *$T_{<\cdot,p>}(H)$ is decidable, where membership in P is denoted by the unary predicate p on H.*

Remark 1: Let $M(H,I,J,P)$ and $M(H',I',J',P')$ be two Rees matrix representations of the same completely simple semigroup C. Assume now that I and J are finite and $T_{\sigma_P}(H)$ is decidable. Then, according to Proposition 2, $T_{<\cdot>}(M(H,I,J,P))$ is decidable. Therefore $T_{<\cdot>}(M(H',I',J',P'))$ is decidable because of the isomorphism of the two semigroups. The finiteness of I and J is equivalent to the finiteness of the number of maximal subgroups in C, which in turn is equivalent to the finiteness of I' and J'. (Let us mention that from the description of isomorphisms between Rees matrix semigroups [P-R] it even follows that $|I| = |I'|$ and $|J| = |J'|$.) Therefore I' and J' are finite. Now if P' is normalized, then $T_{\sigma_{P'}}(H')$ is decidable by Theorem 1. Thus the decidability of $T_{\sigma_P}(H)$ for a **particular** Rees matrix representation of a completely simple semigroup C with a finite number of maximal subgroups implies the decidability of $T_{\sigma_{P'}}(H')$ for **any** Rees matrix representation $M(H',I',J',P')$ of C where P' is normalized. Therefore the criterion given in Theorem 1 does not depend in fact on a particular representation of C as a Rees matrix semigroup.

Note that if I or J is not finite, then the decidability of $T_{<\cdot,p>}(H)$ or the decidability of $T_{<\cdot,a_1,\ldots,a_n>}(H)$ may not imply the decidability of $T_{<\cdot>}(C)$, as is shown in the Example from Section 4. This means, in particular, that Proposition 1 cannot be reversed if I and J are infinite. The next corollary shows that in the case of finite I and J, Proposition 1 cannot be reversed either. This corollary uses the condition (*) defined before Corollary 1.4.

COROLLARY 1.5: *Let a group H and elements $a_1, a_2, \ldots, a_n \in H$ satisfy (*), and let*

$$I \rightleftharpoons \{1,2,\ldots,n\}, \quad J \rightleftharpoons \{1,2\}, \quad P \rightleftharpoons \begin{pmatrix} e & e & e & \ldots & e \\ e & a_1 & a_2 & \ldots & a_n \end{pmatrix},$$
$$C \rightleftharpoons M(H,I,J,P).$$

Then $T_{<\cdot>}(C)$ is undecidable, although $T_{<\cdot>}(H)$ is decidable.

Proof: The undecidability of $T_{<\cdot>}(C)$ follows directly by Theorem 1 from the undecidability of $T_{<\cdot,e,a_1,\ldots,a_n>}(H)$. ∎

Remark 2: Speaking about formulas of signatures $<\cdot, p>$ and $<\cdot, a_1, \ldots, a_n>$ on H, let us mention bialphabetical identities, considered in [Mash]. For a group H with a marked subset P, a bialphabetical identity is an expression of the kind

$$\forall x_1 \ldots \forall x_k \forall y_1 \in P \ldots \forall y_m \in P(u(x_1, \ldots, y_m) = v(x_1, \ldots, y_m)),$$

where u and v are $<\cdot>$ terms on $x_1, \ldots, x_k, y_1, \ldots, y_m$, and $\forall y_i \in P$ means **for every y_i from the set P.**

Such an expression is, obviously, equivalent on H to the formula

$$\forall x_1, \ldots, \forall x_k \forall y_1, \ldots, \forall y_m((\bigwedge_{i=1}^{m} p(y_i)) \to (u(x_1, \ldots, y_m) = v(x_1, \ldots, y_m))),$$

where p is the unary predicate for membership in P.

If P is a finite set $\{a_1, \ldots, a_n\}$, this expression is equivalent on H to the set of identities with fixed points

$$\forall x_1, \ldots, \forall x_k(u(x_1, \ldots, x_k, a_{i_1}, \ldots, a_{i_m}) = v(x_1, \ldots, x_k, a_{i_1}, \ldots, a_{i_m}))$$

for all the sequences $(i_1, \ldots, i_m)$ over $\{1, \ldots, n\}$.

In [Mash] the correspondence between bialphabetical identities of groups of finite exponent with marked subsets and completely simple semigroup identities was established, investigated and used in the finite basis problem.

Now let us discuss the results of Section 3 of this paper. This section concerns the general case, when I and J are not necessarily finite. First of all let us define the class of all two-sorted algebraic systems $D(i_0, j_0)$, which seems to be very natural, but — as far as I know — has never been considered in the literature.

For every $i_0 \in I$, $j_0 \in J$ let $D(i_0, j_0)$ be the two-sorted algebraic system with basic sets H and $I \times J$ of the signature $<\cdot, \circ, \pi>$, where $\cdot$ denotes multiplication on H; $\circ$ denotes multiplication on $I \times J$, defined by $(i_1, j_1) \circ (i_2, j_2) = (i_1, j_2)$; π is the symbol of the unary function that maps the set $I \times J$ into H according to the rule

$$\pi((i, j)) = (p_{ji_0})^{-1} \cdot p_{ji} \cdot (p_{j_0 i})^{-1} \cdot p_{j_0 i_0}.$$

For formulas of the signature $<\cdot, \circ, \pi>$ we will use variables of two kinds: $y_1, y_2, \ldots$ with the domain H and $Y_1, Y_2, \ldots$ with the domain $I \times J$. Let $\Delta \rightleftharpoons \{D(i_0, j_0) \mid i_0 \in I, j_0 \in J\}$.

The following Proposition gives a sufficient condition for decidability of $T_{<\cdot>}(C)$ in terms of the two-sorted system $D(i_0, j_0)$.

PROPOSITION 5: *If $T_{<\cdot,\circ,\pi>}(D(i_0,j_0))$ is decidable for some $i_0 \in I$, $j_0 \in J$, then $T_{<\cdot>}(C)$ is decidable.*

The condition given by Proposition 5 turns out to be necessary for decidability of $T(C)$ with the constant $e(i_0, j_0)$, as the following shows:

PROPOSITION 6: *If $T_{<\cdot,\circ,\pi>}(D(i_0,j_0))$ is undecidable for some $i_0 \in I$, $j_0 \in J$, then $T_{<\cdot,e(i_0,j_0)>}(C)$ is undecidable.*

Let us note that Propositions 5 and 6 are based on Lemmas 3 and 4, which are interesting on their own too. The question whether the decidability of $T_{<\cdot,\circ,\pi>}(D(i_0,j_0))$ is equivalent to the decidability of $T_{<\cdot>}(C)$ is still open in the general case (in the case of a finite sandwich-matrix it is equivalent).

The criterion for decidability of $T_{<\cdot>}(C)$ for the general case is given by the following Theorem.

THEOREM 2: *Let I and J be arbitrary index sets and P be a $J \times I$ matrix over a group H. Then for $C \simeq M(H, I, J, P)$, the following conditions are equivalent:*

1. *$T_{<\cdot>}(C)$ is decidable.*
2. *$T_{<\cdot,\circ,\pi>}(\Delta)$ is decidable, where Δ is the class of all two-sorted systems $D(i_0, j_0)$ for $i_0 \in I$, $j_0 \in J$.*

Remark 3: Let $M(H, I, J, P)$ and $M(H', I', J', P')$ be two Rees matrix representations of the same completely simple semigroup, and let Δ, Δ' be their classes of two-sorted systems, respectively. Assume that $T_{<\cdot,\circ,\pi>}(\Delta)$ is decidable. Then $T_{<\cdot>}(M(H, I, J, P))$ is decidable by Theorem 2. But

$$M(H', I', J', P') \simeq M(H, I, J, P)$$

and therefore $T_{<\cdot>}(M(H', I', J', P'))$ is decidable. Hence $T_{<\cdot,\circ,\pi>}(\Delta')$ is decidable by Theorem 2. Thus the decidability of $T_{<\cdot,\circ,\pi>}(\Delta)$ for a **particular** Rees matrix representation of a completely simple semigroup C implies the decidability of $T_{<\cdot,\circ,\pi>}(\Delta')$ for **any** Rees matrix representation of C. Therefore the criterion given in Theorem 2 does not depend in fact on a particular representation of C as a Rees matrix semigroup.

Note that using the description of isomorphisms between Rees matrix semigroups [P-R] allows us to construct a bijection $f\colon \Delta \longrightarrow \Delta'$ such that every $D(i_0, j_0)$ from Δ is isomorphic (as a two-sorted system) to its image $f(D(i_0, j_0))$

from Δ'. That would give another proof of the fact that $T_{<\cdot,\circ,\pi>}(\Delta)$ is decidable iff $T_{<\cdot,\circ,\pi>}(\Delta')$ is decidable. But in order to involve the elementary theory of C (i.e., $T_{<\cdot>}(M(H,I,J,P))$) we still need to use Theorem 2.

In this paper we will prove only one corollary of Theorem 2. That will be Theorem 3 from Section 4, which gives an example of a completely simple semigroup C_0 over the group $H_3 \rightleftharpoons \{a,\ a^2,\ e\}$ with infinite index sets $I \rightleftharpoons J \rightleftharpoons \mathbf{N} \cup \{0\}$ and with undecidable elementary theory. This example shows that even in the case of a finite group Theorem 1 still needs the condition of finiteness of I and J.

Let us mention that among other corollaries of Theorem 2 are statements about relatively free completely simple semigroups, announced in [R92] and [R93], but they lie outside the framework of the present paper.

2. Proofs of statements directly involving the structure group

First, we give the exact definition of the term **exact interpretation** used informally in the previous sections.

Definition: For arbitrary signatures σ and σ_1 and for algebraic systems A and B of the signatures σ and σ_1 respectively, we say that

$T_\sigma(A)$ is **exactly interpretable** in $T_{\sigma_1}(B)$,

iff for each formula $\varphi \in L_\sigma$ one can effectively construct a formula $\varphi^* \in L_{\sigma_1}$ such that

$\varphi \in T_\sigma(A)$ iff $\varphi^* \in T_{\sigma_1}(B)$.

In this case we say also that

A is exactly interpretable in B.

Then the undecidability of $T_{\sigma_1}(B)$ follows from the undecidability of $T_\sigma(A)$.

LEMMA 1: *Let*

$$\alpha_1(x,z) \rightleftharpoons (x \cdot z = x) \wedge (z \cdot x = x).$$

Then for any $i_0 \in I$, $j_0 \in J$:

the formula $\alpha_1(x, e(i_0, j_0))$ is valid on C

iff $x \in H(i_0, j_0)$.

Proof: Let $x = (h,i,j)$. The validity of $\alpha_1(x, e(i_0,j_0))$ on C means that

$$(h,i,j) \cdot ((p_{j_0 i_0})^{-1}, i_0, j_0) = (h,i,j)$$

and

$$((p_{j_0 i_0})^{-1}, i_0, j_0) \cdot (h,i,j) = (h,i,j).$$

According to the definition of multiplication on C,

$$(h,i,j)\cdot((p_{j_0i_0})^{-1},i_0,j_0)=(h\cdot p_{ji_0}\cdot(p_{j_0i_0})^{-1},i,j_0),$$
$$((p_{j_0i_0})^{-1},i_0,j_0)\cdot(h,i,j)=((p_{j_0i_0})^{-1}\cdot p_{j_0i}\cdot h,i_0,j).$$

Therefore, $C\models\alpha_1(x,e(i_0,j_0))$ iff $i=i_0$ and $j=j_0$, i.e., iff $x\in H(i_0,j_0)$. ∎

PROPOSITION 1: *If $T_{<\cdot>}(H)$ is undecidable, then $T_{<\cdot>}(C)$ is undecidable.*

Proof: We interpret H in C using the above formula $\alpha_1(x,z)$. Actually, for every closed prenex formula $\varphi\in L_{<\cdot>}$ let us restrict its quantifiers, moving in φ from left to right and replacing subformulas of the kind $\forall x\xi$ by $\forall x(\alpha_1(x,z)\to\xi)$, and subformulas of the kind $\exists x\xi$ by $\exists x(\alpha_1(x,z)\wedge\xi)$ (for arbitrary formula ξ). We will finally get the formula of the signature $<\cdot>$ with the free variable z. Let us denote it by $\varphi_1(z)$. Now let $e(i_0,j_0)$ be an arbitrary idempotent from C. Then, by Lemma 1, $H(i_0,j_0)\models\varphi$ iff $C\models\varphi_1(e(i_0,j_0))$. But $H\simeq H(i_0,j_0)$ for any $i_0\in I$, $j_0\in J$, and, therefore,

$H\models\varphi$ iff $C\models\forall z((z^2=z)\to\varphi_1(z))$

(it is also possible to use the formula $\exists z((z^2=z)\wedge\varphi_1(z))$ instead of $\forall z((z^2=z)\to\varphi_1(z))$). Hence if $T_{<\cdot>}(H)$ is undecidable, then $T_{<\cdot>}(C)$ is undecidable too. ∎

PROPOSITION 2: *Let I and J be finite sets. If $T_{<\cdot,a_1,\dots,a_n>}(H)$ is decidable, where $a_1,\dots,a_n$ are symbols for all distinct elements of H belonging to P, then $T_{<\cdot>}(C)$ is decidable.*

Proof: Assume that $I=\{1,\dots,M\}$, $J=\{1,\dots,N\}$, and let $T_{<\cdot a_1,\dots,a_n>}(H)$ be decidable. For any $j\in J$, $i\in I$ let us define on H the unary predicate $r_{ji}(x)$, which holds iff $x=p_{ji}$, i.e., iff $x=a_k$ for the corresponding $a_k\in\{a_1,\dots,a_n\}$, which is the element of the j-th row and i-th column of the sandwich-matrix P. The correspondence η: $(i,j)\mapsto k$ can be effectively defined because of the finiteness of the matrix P. Let $\sigma\rightleftharpoons<\cdot,\{r_{ji}\mid 1\le j\le N,1\le i\le M\}>$. Replacing expressions $r_{ji}(x)$ by $x=a_k$ for $k=\eta(i,j)$ in σ-formulas on H gives the exact interpretation of the theory $T_\sigma(H)$ in $T_{<\cdot,a_1,\dots,a_n>}(H)$. Therefore, $T_\sigma(H)$ is decidable. Now let us consider the set I as the algebraic system of the signature σ, where $i_1\cdot i_2\rightleftharpoons i_1$, and where $r_{ji}(i_1)$ holds iff $i_1=i$. Analogously, on the set J let us define $j_1\cdot j_2=j_2$, and define that the predicate $r_{ji}(j_1)$ holds iff $j_1=j$. The theories $T_\sigma(I)$ and $T_\sigma(J)$ are, obviously, decidable. Let $\Pi\rightleftharpoons H\times I\times J$ be the direct product of these three algebraic systems of the signature

σ. Then Π is the algebraic system of the signature σ with the multiplication $(h_1, i_1, j_1) \cdot (h_2, i_2, j_2) = (h_1 \cdot h_2, i_1, j_2)$, and for any $i_0 \in I$, $j_0 \in J$ the predicate $r_{j_0 i_0}(x)$ holds on Π iff $x = (p_{j_0 i_0}, i_0, j_0)$. The theory $T_\sigma(\Pi)$ is decidable because of the decidability of $T_\sigma(H)$, $T_\sigma(I)$ and $T_\sigma(J)$ [M].

We will formally define on Π a new binary operation, corresponding to the multiplication on C. Let

$$\beta(x_1, x_2, x_3) \rightleftharpoons \exists v, t((\bigvee_{\substack{1\le i\le M\\ 1\le j\le N}} r_{ji}(v)) \wedge (x_1 \cdot v \cdot x_2 = x_3) \wedge (t \cdot v = v) \wedge (v \cdot t = v)$$
$$\wedge (x_1 \cdot t = x_1) \wedge (t \cdot x_2 = x_2)).$$

Let x_1, x_2, x_3, v, t be arbitrary elements of Π, where $x_l = (h_l, i_l, j_l)$ for $l = 1, 2, 3$. The formula

$$(\bigvee_{\substack{1\le i\le M\\ 1\le j\le N}} r_{ji}(v)) \wedge (x_1 \cdot v \cdot x_2 = x_3)$$

means that there exist $i_0 \in I$, $j_0 \in J$ such that $v = (p_{j_0 i_0}, i_0, j_0)$ and $h_3 = h_1 \cdot p_{j_0 i_0} \cdot h_2$ in H, $i_3 = i_1, j_3 = j_2$. For x_1, x_2, x_3, v, satisfying these conditions, the formula

$$(t \cdot v = v) \wedge (v \cdot t = v) \wedge (x_1 \cdot t = x_1) \wedge (t \cdot x_2 = x_2)$$

means that $t = (e, i_0, j_0)$ and that $j_0 = j_1$, $i_0 = i_2$. Therefore, if $\beta(x_1, x_2, x_3)$ is true on Π, then $x_3 = (h_1 \cdot p_{j_1 i_2} \cdot h_2, i_1, j_2)$. Conversely, if $x_3 = (h_1 \cdot p_{j_1 i_2} \cdot h_2, i_1, j_2)$, we can take $v = (p_{j_1 i_2}, i_2, j_1)$, $t = (e, i_2, j_1)$, and that will prove the validity of $\beta(x_1, x_2, x_3)$ on Π.

Now let φ be a formula from $L_{<\cdot>}$. Assume, without loss of generality, that quantifier-free subformulas of φ are boolean combinations of atomic formulas of the kinds $x_1 = x_2$, $x_1 \cdot x_2 = x_3$ or their negations, where x-s are variables. We can construct the corresponding formula $\varphi\prime \in L_\sigma$, replacing in φ all atomic formulas of the kind $x_1 \cdot x_2 = x_3$ by $\beta(x_1, x_2, x_3)$. We obtain that

$\varphi \in T_{<\cdot>}(C)$ iff $\varphi\prime \in T_\sigma(\Pi)$.

Therefore, $T_{<\cdot>}(C)$ is decidable. ∎

In the following Lemma we use the formula $\alpha_1(x, z)$ from Lemma 1.

LEMMA 2: *Let*

$$\alpha(x, z) \rightleftharpoons \alpha_1(x, z) \wedge (\exists v)((v^2 = v) \wedge (x \cdot v \cdot z = z)),$$

and let P be a (i_0, j_0)-normalized matrix. Then:

the formula $\alpha(x, e(i_0, j_0))$ *is valid on* C
iff $x = (p_{ji}, i_0, j_0)$ *for* $i \in I$, $j \in J$.

Proof: Let $x = (h, i, j)$. The element $p_{j_0 i_0} \in P$ equals e, and therefore $e(i_0, j_0) = (e, i_0, j_0)$. The validity of $\alpha(x, e(i_0, j_0))$ on C means that $x \in H(i_0, j_0)$ by Lemma 1, i.e., $i = i_0$, $j = j_0$, and that there exists $v = ((p_{j_1 i_1})^{-1}, i_1, j_1)$ such that $x \cdot v \cdot e(i_0, j_0) = e(i_0, j_0)$. But

$$\begin{aligned} x \cdot v \cdot e(i_0, j_0) &= (h, i_0, j_0) \cdot ((p_{j_1 i_1})^{-1}, i_1, j_1) \cdot (e, i_0, j_0) \\ &= (h \cdot p_{j_0 i_1} \cdot (p_{j_1 i_1})^{-1} \cdot p_{j_1 i_0} \cdot e, i_0, j_0) = (h \cdot (p_{j_1 i_1})^{-1}, i_0, j_0), \end{aligned}$$

because $p_{j_0 i_1} = p_{j_1 i_0} = e$ in the (i_0, j_0)-normalized matrix P. Therefore,
$x \cdot v \cdot e(i_0, j_0) = e(i_0, j_0)$ iff $h = p_{j_1 i_1}$, i.e., iff $x = (p_{j_1 i_1}, i_0, j_0)$. ∎

PROPOSITION 3: *Let* P *be an* (i_0, j_0)*-normalized matrix. If* $T_{<\cdot, p>}(H)$ *is undecidable, where* p *is the unary predicate for membership in* P, *then* $T_{<\cdot, e(i_0, j_0)>}(C)$ *is undecidable.*

Proof: Recall the mapping $h \mapsto (h \cdot (p_{j_0 i_0})^{-1}, i_0, j_0)$, which defines the isomorphism of groups H and $H(i_0, j_0)$ (see Section 1). The element $p_{j_0 i_0}$ equals e, and therefore $e(i_0, j_0) = (e, i_0, j_0)$. Hence for any $h \in H$ the image of h is (h, i_0, j_0); in particular, the image of p_{ji} is (p_{ji}, i_0, j_0) for any $i \in I$, $j \in J$.

Now for every closed prenex formula $\varphi \in L_{<\cdot, p>}$ let us restrict its quantifiers using the formula $\alpha_1(x, z)$, as in the proof of Proposition 1. In the resulting formula let us replace all atomic formulas of the kind $p(x)$ by $\alpha(x, z)$. Finally, we will obtain a formula from $L_{<\cdot>}$ whose only free variable is z. Let us denote this formula by $\varphi_2(z)$. From its construction and Lemmas 1 and 2 it follows that
$H \models \varphi$ iff $C \models \varphi_2(e(i_0, j_0))$.
Hence if $T_{<\cdot, p>}(H)$ is undecidable, then $T_{<\cdot, e(i_0, j_0)>}(C)$ is undecidable too. ∎

The following proposition is a simple fact from model theory. Here σ is an arbitrary signature.

PROPOSITION 4: *Let* M *be an algebraic system of a signature* σ; *suppose that its theory* $T_\sigma(M)$ *is decidable; suppose also that* $\gamma(x)$ *is a formula from* L_σ *such that* $A \rightleftharpoons \{x \in M \mid M \models \gamma(x)\}$ *is finite, say* $A = \{a_1, \ldots, a_n\}$. *Let* $\sigma^* \rightleftharpoons \sigma \cup \{a_1, \ldots, a_n\}$ *be the signature obtained by adding the constants* $a_1, \ldots, a_n$ *to* σ. *Then* $T_{\sigma^*}(M)$ *is decidable.*

Proof: Our aim is to interpret (exactly) the theory $T_{\sigma^*}(M)$ in $T_\sigma(M)$. Let τ be any permutation on $\{1, \ldots, n\}$. Assume first that τ satisfies the following condition:

(**) for every formula $\psi(x_1,\ldots,x_n) \in L_\sigma$ the formula $\psi(a_1,\ldots,a_n)$ is valid on M iff the formula $\psi(a_{\tau(1)},\ldots,a_{\tau(n)})$ is valid on M.

Then let us define

$$\psi_\tau(x_1,\ldots,x_n) \rightleftharpoons (x_1 = x_1).$$

Assume now that τ does not satisfy (**), and let $\psi_\tau(x_1,\ldots,x_n)$ be a formula from L_σ with only free variables $x_1,\ldots,x_n$ such that:

(***) $\psi_\tau(a_1,\ldots,a_n)$ is valid on M, but $\psi_\tau(a_{\tau(1)},\ldots,a_{\tau(n)})$ is not valid on M.

Let S_n be the group of all permutations on $\{1,\ldots,n\}$, and let

$$\delta(x_1,\ldots,x_n) \rightleftharpoons (\bigwedge_{i=1}^{n}\gamma(x_i)) \wedge (\bigwedge_{1\le i<j\le n}(x_i \neq x_j)) \wedge (\bigwedge_{\tau\in S_n}\psi_\tau(x_1,\ldots,x_n)).$$

The formula $\delta(x_1,\ldots,x_n)$ belongs to L_σ, and it is valid on M iff

$$x_1 = a_{\tau(1)},\ldots,x_n = a_{\tau(n)},$$

where a permutation $\tau \in S_n$ satisfies (**).

Now for every formula $\varphi(x_1,\ldots,x_n) \in L_\sigma$ with the only free variables $x_1,\ldots,x_n$ we can construct the corresponding closed formula $\hat{\varphi} \in L_\sigma$ as follows:

$\hat{\varphi} \rightleftharpoons \exists x_1,\ldots,x_n(\delta(x_1,\ldots,x_n) \wedge \varphi(x_1,\ldots,x_n))$.

Note that the following condition holds:

(****) the formula $\varphi(a_1,\ldots,a_n)$ is valid on M iff the closed formula $\hat{\varphi}$ of the signature σ is valid on M.

Now let θ be a closed formula of the signature σ^*. Assume, without loss of generality, that θ does not contain the symbols $x_1,\ldots,x_n$. Let us replace the symbols $a_1,\ldots,a_n$ in θ by $x_1,\ldots,x_n$, respectively, and denote the resulting formula by $\varphi_\theta(x_1,\ldots,x_n)$. Then $\varphi_\theta(x_1,\ldots,x_n) \in L_\sigma$ and θ is $\varphi_\theta(a_1,\ldots,a_n)$. We obtain by (****) that $\varphi_\theta(a_1,\ldots,a_n) \in T_{\sigma^*}(M)$ iff $\hat{\varphi}_\theta \in T_\sigma(M)$. Therefore, the decidability of $T_{\sigma^*}(M)$ follows from the decidability of $T_\sigma(M)$. Proposition 4 is proved. ∎

Remark 4: We proved that if M and A satisfy the conditions of Proposition 4, then there **exists** an algorithm that transforms every closed formula from $L_{\sigma\cup\{a_1,\ldots,a_n\}}$ to a corresponding closed formula from L_σ. This algorithm uses $\mid S_n \mid = n!$ formulas $\psi_\tau(x_1,\ldots,x_n)$, where some of them satisfy (***) and others coincide with the formula $x_1 = x_1$ (the last case holds iff τ satisfies (**)). We know only that there **exists** a finite set of such formulas $\{\psi_\tau \mid \tau \in S_n\}$, but we do not give an **effective** way to find them. Therefore, the proof of this Proposition is not constructive, and the same is true for the proofs (1) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (2) of the following Theorem.

THEOREM 1: *Let I and J be finite sets and P be a normalized $J \times I$ matrix over a group H. Then for $C \simeq M(H, I, J, P)$, the following three conditions are equivalent:*

1. *$T_{<\cdot>}(C)$ is decidable.*
2. *$T_{<\cdot, a_1, \ldots, a_n>}(H)$ is decidable, where $a_1, \ldots, a_n$ are symbols for all distinct elements of H belonging to P.*
3. *$T_{<\cdot, p>}(H)$ is decidable, where membership in P is denoted by the unary predicate p on H.*

Proof: (2) $\Longrightarrow$ (1) by Proposition 2.

(3) $\Longrightarrow$ (2) by Proposition 4, when we take:

$\sigma \rightleftharpoons <\cdot, p>$; $M \rightleftharpoons H$; A is the set of all distinct elements of H belonging to P; $\gamma(x) \rightleftharpoons p(x)$.

(1) $\Longrightarrow$ (3). Let $T_{<\cdot>}(C)$ be decidable, and let $i_0 \in I$, $j_0 \in J$ be such that P is (i_0, j_0)-normalized. Then $T_{<\cdot, e(i_0, j_0)>}(C)$ is decidable by Proposition 4, when we take:

$\sigma \rightleftharpoons <\cdot>$; $M \rightleftharpoons C$; $A \rightleftharpoons E$, the set of all idempotents of C; $\gamma(x) \rightleftharpoons x^2 = x$. Now the decidability of $T_{<\cdot, p>}(H)$ follows from the decidability of $T_{<\cdot, e(i_0, j_0)>}(C)$ by Proposition 3. ∎

3. Proofs of statements involving two-sorted systems

In this section we deal with the general case of a completely simple semigroup C, presented by a Rees matrix semigroup $M(H, I, J, P)$ over a group H, where I and J can be infinite, P is not necessarily normalized. The following lemma gives an exact interpretation of C in the two-sorted system $D(i_0, j_0)$ for every $i_0 \in I$, $j_0 \in J$.

LEMMA 3: *There exists a recursive mapping $L_{<\cdot>} \mapsto L_{<\cdot, \circ, \pi>}$, giving for every closed formula $\psi \in L_{<\cdot>}$ a corresponding closed formula $\psi_1 \in L_{<\cdot, \circ, \pi>}$, in such a way that, for every $i_0 \in I$, $j_0 \in J$:*

$C \models \psi$ iff $D(i_0, j_0) \models \psi_1$.

Proof: We will effectively construct this mapping. Let ψ be a formula from $L_{<\cdot>}$. Assume without loss of generality that $x_1, \ldots, x_n$ are all the variables occurring in ψ, and that quantifier-free subformulas of ψ are boolean combinations of atomic formulas $x_{k_1} = x_{k_2}$, $x_{k_1} \cdot x_{k_2} = x_{k_3}$ or their negations. To each x_k we match the pair of variables y_k, Y_k; the idea is that if x_k stands for (h, i, j), then y_k stands for $h \in H$ and Y_k stands for $(i, j) \in I \times J$. Let us replace every quantifier $\exists x_k$ by $\exists y_k \exists Y_k$, every quantifier $\forall x_k$ by $\forall y_k \forall Y_k$, every formula $x_{k_1} = x_{k_2}$ by $(y_{k_1} = y_{k_2}) \wedge (Y_{k_1} = Y_{k_2})$, and let us replace every formula $x_{k_1} \cdot x_{k_2} = x_{k_3}$ by

$$(Y_{k_1} \circ Y_{k_2} = Y_{k_3}) \wedge (y_{k_1} \cdot \pi(Y_{k_2} \circ Y_{k_1}) \cdot y_{k_2} = y_{k_3}).$$

Denote by ψ_1 the formula from $L_{<\cdot,\circ,\pi>}$ obtained by this algorithm.

Assume now that ψ is a closed formula. Then ψ_1 is closed too, and for any $i_0 \in I$, $j_0 \in J$ the validity of ψ on the semigroup $M(H, I, J, P(i_0, j_0))$ is equivalent to the validity of ψ_1 on $D(i_0, j_0)$. This equivalence follows immediately from the presentation of $M(H, I, J, P(i_0, j_0))$ by triples. The semigroup $M(H, I, J, P(i_0, j_0))$ is isomorphic to $M(H, I, J, P)$, i.e., to C (this was mentioned in Section 1; for the proof see [C-P]). Therefore, $C \models \psi$ iff $D(i_0, j_0) \models \psi_1$. Lemma 3 is proved. ∎

PROPOSITION 5: *If $T_{<\cdot,\circ,\pi>}(D(i_0, j_0))$ is decidable for some $i_0 \in I$, $j_0 \in J$, then $T_{<\cdot>}(C)$ is decidable.*

Proof: This follows directly from Lemma 3. ∎

In the following Lemma 4 we will formally define the system $D(i_0, j_0)$ in C, using one-parameter formulas from $L_{<\cdot>}$. The proof will generalize the proof of Proposition 3.

LEMMA 4: *There exists a recursive mapping $L_{<\cdot,\circ,\pi>} \mapsto L_{<\cdot>}$, giving for every closed formula $\varphi \in L_{<\cdot,\circ,\pi>}$ a corresponding formula $\varphi_1(z) \in L_{<\cdot>}$, in such a way that, for every $i_0 \in I$, $j_0 \in J$:*

$D(i_0, j_0) \models \varphi$ iff $C \models \varphi_1(e(i_0, j_0))$.

Proof: Let us effectively construct this mapping. We will write down a set Φ of 7 formulas from $L_{<\cdot>}$, containing the parameter z and such that for every $i_0 \in I$, $j_0 \in J$ the set of 7 formulas obtained from Φ by substitution of $e(i_0, j_0)$ for z defines in C a two-sorted system isomorphic to $D(i_0, j_0)$. Let

$$\alpha_1(x, z) \rightleftharpoons (x \cdot z = x) \wedge (z \cdot x = x)$$

be the formula from the Lemma 1, and let

$$\alpha_2(x) \rightleftharpoons x^2 = x.$$

Let us fix arbitrary $i_0 \in I$, $j_0 \in J$. Then the formula $\alpha_1(x, e(i_0, j_0))$ picks out from C the group $H(i_0, j_0)$, and the formula $\alpha_2(x)$ picks out from C the set E of all idempotents. The mapping $U(h) \rightleftharpoons (h \cdot (p_{j_0 i_0})^{-1}, i_0, j_0)$ defines an isomorphism of H onto $H(i_0, j_0)$ (see Section 1). It is easy to see that the mapping $V((i,j)) \rightleftharpoons e(i,j)$ defines a one-one correspondence between $I \times J$ and E. Let

$$\alpha_3(x_1, x_2, z) \rightleftharpoons x_1 \cdot (z \cdot x_2 \cdot z) = z.$$

It is easy to check for the following table, that any $y_1, y_2, y_3 \in H$, any $Y_1, Y_2, Y_3 \in I \times J$ and the corresponding

$$u_k \rightleftharpoons U(y_k) = (y_k \cdot (p_{j_0 i_0})^{-1}, i_0, j_0), \quad v_k \rightleftharpoons V(Y_k) = ((p_{j_k i_k})^{-1}, i_k, j_k)$$

for $1 \le k \le 3$ satisfy the condition:

Every equality from the left column holds on $D(i_0, j_0)$ iff the corresponding equality from the right column holds on C.

$y_1 = y_2$	$u_1 = u_2$
$y_1 \cdot y_2 = y_3$	$u_1 \cdot u_2 = u_3$
$Y_1 = Y_2$	$v_1 = v_2$
$Y_1 \circ Y_2 = Y_3$	$(v_1 \cdot v_2) \cdot v_3 = v_3 \cdot (v_1 \cdot v_2)$
$\pi(Y_1) = y_1$	$\alpha_3(u_1, v_1, e(i_0, j_0))$; that is $u_1 \cdot (e(i_0, j_0) \cdot v_1 \cdot e(i_0, j_0)) = e(i_0, j_0)$

Consider, for example, the last row of the table. Let us take arbitrary $y_1 \in H$, $Y_1 = (i_1, j_1) \in I \times J$, and let

$$u_1 \rightleftharpoons U(y_1) = (y_1 \cdot (p_{j_0 i_0})^{-1}, i_0, j_0) \in C,$$
$$v_1 \rightleftharpoons V(Y_1) = ((p_{j_1 i_1})^{-1}, i_1, j_1) \in C.$$

Then, according to the definition of multiplication in C,

$$\begin{aligned} & u_1 \cdot (e(i_0, j_0) \cdot v_1 \cdot e(i_0, j_0)) \\ &= (y_1 \cdot (p_{j_0 i_0})^{-1}, i_0, j_0) \cdot ((p_{j_0 i_0})^{-1}, i_0, j_0) \cdot ((p_{j_1 i_1})^{-1}, i_1, j_1) \cdot ((p_{j_0 i_0})^{-1}, i_0, j_0) \\ &= (y_1 \cdot (p_{j_0 i_0})^{-1} \cdot p_{j_0 i_0} \cdot (p_{j_0 i_0})^{-1} \cdot p_{j_0 i_1} \cdot (p_{j_1 i_1})^{-1} \cdot p_{j_1 i_0} \cdot (p_{j_0 i_0})^{-1}, i_0, j_0) \\ &= (y_1 \cdot (p_{j_0 i_0})^{-1} \cdot p_{j_0 i_1} \cdot (p_{j_1 i_1})^{-1} \cdot p_{j_1 i_0} \cdot (p_{j_0 i_0})^{-1}, i_0, j_0), \end{aligned}$$

and therefore the formula $u_1 \cdot (e(i_0, j_0) \cdot v_1 \cdot e(i_0, j_0)) = e(i_0, j_0)$ on C means that the equality

$$(y_1 \cdot (p_{j_0 i_0})^{-1} \cdot p_{j_0 i_1} \cdot (p_{j_1 i_1})^{-1} \cdot p_{j_1 i_0} \cdot (p_{j_0 i_0})^{-1}, i_0, j_0) = ((p_{j_0 i_0})^{-1}, i_0, j_0)$$

holds on C.

But that is equivalent to the equality $y_1 = (p_{j_1 i_0})^{-1} \cdot p_{j_1 i_1} \cdot (p_{j_0 i_1})^{-1} \cdot p_{j_0 i_0}$ on H. The last equality means exactly that $\pi(Y_1) = y_1$ in $D(i_0, j_0)$, q.e.d.

Now let us consider symbols u_1, u_2, u_3 as variables with the domain $H(i_0, j_0) \subseteq C$, and symbols v_1, v_2, v_3 as variables with the domain $E \subseteq C$. Then the formulas from the right column of the table define on $H(i_0, j_0)$ and E the predicates corresponding to the signature $< \cdot, \circ, \pi >$, where $H(i_0, j_0)$ and E are considered as two basic sets of a two-sorted system. This system is isomorphic to $D(i_0, j_0)$. Let Φ be the set of formulas consisting of α_1, α_2, α_3 and of the 4 other right-side formulas from the table. It satisfies the conditions mentioned at the beginning of the proof.

The last step of the proof of this lemma is standard, but nevertheless let us write it down. Let φ be a formula from $L_{<\cdot,\circ,\pi>}$. Assume, without loss of generality, that $y_1, \ldots, y_n, Y_1, \ldots, Y_m$ are all the variables occurring in φ, and that quantifier-free subformulas of φ are boolean combinations of atomic formulas of the kinds from the left column of the table or their negations. To each y_t we match the variable u_t, to each Y_k the variable v_k, and replace in φ all the $\exists y_t$, $\forall y_t$, $\exists Y_k$, $\forall Y_k$ by $\exists u_t$, $\forall u_t$, $\exists v_k$, $\forall v_k$ respectively. In the quantifier-free subformulas we replace atomic formulas of the kind $\pi(Y_k) = y_t$ by $\alpha_3(u_t, v_k, z)$, and we replace other atomic formulas by the corresponding formulas from the right column of the table. After that we restrict the quantifiers, using $\alpha_1(u_t, z)$ to restrict by $H(i_0, j_0)$ the quantifiers on u_t-s, and using $\alpha_2(v_k)$ to restrict by E the quantifiers on v_k-s. The resulting formula belongs to $L_{<\cdot>}$, it does not depend on i_0, j_0 and has the free variable z.

If φ is a closed formula, then the resulting formula has no other free variables, and we denote it by $\varphi_1(z)$. From the construction of $\varphi_1(z)$ it follows that:

$D(i_0, j_0) \models \varphi$ iff $C \models \varphi_1(e(i_0, j_0))$.

Lemma 4 is proved. ∎

Proposition 6: *If $T_{<\cdot,\circ,\pi>}(D(i_0, j_0))$ is undecidable for some $i_0 \in I$, $j_0 \in J$, then $T_{<\cdot,e(i_0,j_0)>}(C)$ is undecidable.*

Proof: This follows directly from Lemma 4. ∎

Remark 5: If P is an (i_0, j_0)-normalized matrix, the function π on $D(i_0, j_0)$ turns out to be as follows:

$$\pi((i,j)) = (p_{j i_0})^{-1} \cdot p_{ji} \cdot (p_{j_0 i})^{-1} \cdot p_{j_0 i_0} = p_{ji}, \quad \text{because } p_{j i_0} = p_{j_0 i} = p_{j_0 i_0} = e.$$

Therefore, the formula $\exists Y(\pi(Y) = y)$ defines the unary predicate on the basic set H of $D(i_0, j_0)$, corresponding to membership in P (we denote this predicate by p — see previous sections). Hence the group H itself, considered in the signature $< \cdot, p >$, is exactly interpretable in $D(i_0, j_0)$ (note that if P is not normalized, that may be impossible for any $i_0 \in I, j_0 \in J$). Now, using the exact interpretation of $D(i_0, j_0)$ in C with the constant $e(i_0, j_0)$ in the signature given by Lemma 4, we will finally get the exact interpretation of H in C for the mentioned signatures. This provides another proof for Proposition 3, although the algorithm obtained here is more complicated than the original one.

THEOREM 2: *Let I and J be arbitrary index sets and P be a $J \times I$ matrix over a group H. Then for $C \simeq M(H, I, J, P)$, the following conditions are equivalent:*

1. *$T_{<\cdot>}(C)$ is decidable.*
2. *$T_{<\cdot,\circ,\pi>}(\Delta)$ is decidable, where Δ is the class of all two-sorted systems $D(i_0, j_0)$ for $i_0 \in I$, $j_0 \in J$.*

Proof: Let $T_{<\cdot>}(C)$ be decidable, and let φ be a closed formula from $L_{<\cdot,\circ,\pi>}$. Then for $\varphi_1(z)$ from Lemma 4 we have:

$\Delta \models \varphi$ iff $C \models \forall z((z^2 = z) \to \varphi_1(z))$.

Therefore $T_{<\cdot,\circ,\pi>}(\Delta)$ is decidable.

Conversely, let $T_{<\cdot,\circ,\pi>}(\Delta)$ be decidable, and let ψ be a closed formula from $L_{<\cdot>}$. Then for ψ_1 from Lemma 3 we have:

$C \models \psi$ iff $\Delta \models \psi_1$.

Therefore $T_{<\cdot>}(C)$ is decidable, and Theorem 2 is proved. ∎

4. An example of a completely simple semigroup with a finite structure group and with undecidable elementary theory

Let

$$H_3 \rightleftharpoons \{a,\ a^2,\ e\} \quad \text{with } a^3 = e,\ I \rightleftharpoons J \rightleftharpoons \mathbf{N} \cup \{0\},$$

and let $K \rightleftharpoons \{m(1), \ldots, m(i), \ldots\}$ be a recursively enumerable non-recursive subset of $\mathbf{N}$.

Now let

$$P \rightleftharpoons \begin{pmatrix} e & e & \dots & e & \dots \\ e & a & \dots & a & \dots \\ \dots & \dots & \dots & \dots & \dots \\ e & a & \dots & a & \dots \\ e & a & \dots & a^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ e & a & \dots & a^2 & \dots \\ e & a^2 & \dots & a^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ e & a^2 & \dots & a^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

where

$$p_{ji} = \begin{cases} e, & \text{if } j = 0 \text{ or } i = 0, \\ a, & \text{if } i > 0 \text{ and } 0 < j \leq m(i), \\ a^2, & \text{if } i > 0 \text{ and } j > m(i). \end{cases}$$

Let

$$C_0 \rightleftharpoons M(H_3, I, J, P).$$

The aim of this section is to prove that $T_{<\cdot>}(C_0)$ is undecidable.

In the next lemma we will use the two-sorted system $D(i_0, j_0)$ for $i_0 = 0 \in I$, $j_0 = 0 \in J$. The sandwich-matrix P is $(0,0)$-normalized; therefore, according to Remark 5 from Section 3, the function π on $D(0,0)$ is as follows: $\pi((i,j)) = p_{ji}$.

Lemma E 1: *Let*

$$\beta_1(y) \rightleftharpoons y \neq e \wedge (\exists Y_1)(\pi(Y_1) = y \wedge$$
$$(\forall Y_2)((Y_1 \circ Y_2 = Y_1) \rightarrow (\pi(Y_2) = e \vee \pi(Y_2) = y))).$$

Then $\beta_1(y)$ is valid on $D(0,0)$ iff $y = a$.

Proof: For any $Y_1 = (i_1, j_1)$, $Y_2 = (i_2, j_2)$ the equality $Y_1 \circ Y_2 = Y_1$ means that $j_1 = j_2$, i.e., that $\pi(Y_1) = p_{j_1 i_1}$ and $\pi(Y_2) = p_{j_2 i_2}$ belong to the same j_1-th row of P. Therefore, informally speaking, the validity of $\beta_1(y)$ on $D(0,0)$ means that the following condition holds:

(E*) $y \neq e$ and there exists $j_1 \in J$ such that the j_1-th row of P contains y at least once and does not contain elements that differ from e and y.

Suppose now that $y = a$. Then we can take $j_1 = 1$. There is no a^2 in row number 1 of P, because $m(i_2) > 0$ for any $i_2 > 0$. Therefore, (E*) holds.

Conversely, suppose that (E*) holds. Then $y \neq a^2$, because it is always possible to find i_2 such that $m(i_2)$ will be bigger than j_1, and therefore p_{j_1, i_2} will be equal to a. Hence, $y = a$, and the lemma is proved. ∎

LEMMA E 2: *For any $m \geq 1$ let*

$$\gamma_m \rightleftharpoons \exists y(\beta_1(y) \wedge (\exists Y_1, \dots, Y_m)((\pi(Y_1) = y) \wedge$$
$$(\bigwedge_{1 \leq j_1 < j_2 \leq m} ((Y_{j_1} \neq Y_{j_2}) \wedge (Y_{j_1} \circ Y_{j_2} = Y_{j_2}) \wedge (\pi(Y_{j_2}) = y))) \wedge$$
$$(\forall Y_{m+1})((\bigwedge_{1 \leq j \leq m} ((Y_j \neq Y_{m+1}) \wedge (Y_j \circ Y_{m+1} = Y_{m+1}))) \to (\pi(Y_{m+1}) \neq y))))).$$

Then $D(0,0) \models \gamma_m$ iff $m \in K$.

Proof: For any $Y_1 = (i_1, j_1)$, $Y_2 = (i_2, j_2)$ the equality $Y_1 \circ Y_2 = Y_2$ means that $i_1 = i_2$, i.e., that $\pi(Y_1) = p_{j_1 i_1}$ and $\pi(Y_2) = p_{j_2 i_2}$ belong to the same i_1-th column of P. Therefore, informally speaking, the validity of γ_m on $D(0,0)$ means, according to Lemma E 1, that the following condition holds:

There exists $i_1 \in I$ such that the i_1-th column of P contains a at m different places, but it does not contain a at $m+1$ different places.

According to the definition of P, for every $i_1 > 0$ the i_1-th column of P contains a at $m(i_1)$ different places exactly. Therefore, γ_m means that there exists $i_1 > 0$ such that $m = m(i_1)$, which means $m \in K$. The lemma is proved. ∎

PROPOSITION E 1: *$T_{<\cdot,\circ,\pi>}(D(0,0))$ is undecidable.*

Proof: For any $m \geq 1$ the formula γ_m defined in Lemma E 2 is a closed formula from $L_{<\cdot,\circ,\pi,e>}$, and it belongs to $T_{<\cdot,\circ,\pi,e>}(D(0,0))$ iff $m \in K$. The set K is not recursive, therefore $T_{<\cdot,\circ,\pi,e>}(D(0,0))$ is undecidable. Hence $T_{<\cdot,\circ,\pi>}(D(0,0))$ is undecidable too, because every atomic formula $y = e$ on $D(0,0)$ can be replaced by $y^2 = y$. ∎

The next lemma gives a property that specifies the system $D(0,0)$ in the class $\Delta \rightleftharpoons \{D(i_0, j_0) \mid i_0 \in I, j_0 \in J\}$. More precisely, this lemma establishes the negation of such a property.

LEMMA E 3: *For every $i_0 \in I$, $j_0 \in J$, where $i_0 \neq 0$ or $j_0 \neq 0$, there exist $i_1 \in I \setminus i_0$, $j_1 \in J \setminus j_0$ such that in the matrix $P(i_0, j_0)$ the element of the j_1-th row and i_1-th column equals e.*

Proof: Let q_{ji} denote the element of the j-th row and i-th column of $P(i_0, j_0)$. Then, according to the definition of $P(i_0, j_0)$ (Section 1),

$q_{ji} = (p_{j i_0})^{-1} \cdot p_{ji} \cdot (p_{j_0 i})^{-1} \cdot p_{j_0 i_0}$.

CASE 1: If $i_0 = 0$, $j_0 > 0$, let us take any $0 < i_1 \in I$ such that $m(i_1) > j_0$, and let us take $j_1 \rightleftharpoons m(i_1)$. Then $j_1 \in J \setminus \{0, j_0\}$. Hence we get:

$$\begin{array}{lll} i_0 = 0 & \Longrightarrow & p_{ji_0} = p_{j_0 i_0} = e, \\ (0 < i_1) \wedge (0 < j_0 < m(i_1)) & \Longrightarrow & p_{j_0 i_1} = a, \\ (0 < i_1) \wedge (0 < j_1 = m(i_1)) & \Longrightarrow & p_{j_1 i_1} = a. \end{array}$$

Therefore,

$$q_{j_1 i_1} = e^{-1} \cdot a \cdot a^{-1} \cdot e = e.$$

CASE 2: If $i_0 > 0$, $j_0 = 0$, let us take any $0 < i_1 \in I \setminus \{i_0\}$ and $j_1 \rightleftharpoons 1$. Then:

$$\begin{array}{lll} j_0 = 0 & \Longrightarrow & p_{j_0 i_0} = p_{j_0 i_1} = e, \\ (0 < i_1) \wedge (j_1 = 1) & \Longrightarrow & p_{j_1 i_1} = a, \\ (0 < i_0) \wedge (j_1 = 1) & \Longrightarrow & p_{j_1 i_0} = a. \end{array}$$

Therefore,

$$q_{j_1 i_1} = a^{-1} \cdot a \cdot e^{-1} \cdot e = e.$$

CASE 3: If $i_0 > 0$, $j_0 > 0$, let us consider two subcases:

(a) $j_0 \leq m(i_0)$. Let us take $i_1 \in I \setminus \{i_0\}$ such that $j_0 \leq m(i_1)$, and let $j_1 \rightleftharpoons 0$. Then:

$$\begin{array}{lll} j_1 = 0 & \Longrightarrow & p_{j_1 i_0} = p_{j_1 i_1} = e, \\ 0 < j_0 \leq m(i_1) & \Longrightarrow & p_{j_0 i_1} = a, \\ 0 < j_0 \leq m(i_0) & \Longrightarrow & p_{j_0 i_0} = a. \end{array}$$

Therefore,

$$q_{j_1 i_1} = e^{-1} \cdot e \cdot a^{-1} \cdot a = e.$$

(b) $j_0 > m(i_0)$. Let us take $i_1 \rightleftharpoons 0$, $j_1 > j_0$. Then:

$$\begin{array}{lll} i_1 = 0 & \Longrightarrow & p_{j_0 i_1} = p_{j_1 i_1} = e, \\ j_0 > m(i_0) & \Longrightarrow & p_{j_0 i_0} = a^2, \\ j_1 > j_0 > m(i_0) & \Longrightarrow & p_{j_1 i_0} = a^2. \end{array}$$

Therefore,

$$q_{j_1 i_1} = (a^2)^{-1} \cdot e \cdot e^{-1} \cdot a^2 = e. \qquad \blacksquare$$

Now let us specify the system $D(0,0)$ in the class Δ by an elementary formula. As in the proof of Lemma E 3, for fixed $i_0 \in I$, $j_0 \in J$ we will denote by q_{ji} the element of the j-th row and i-th column of the matrix $P(i_0, j_0)$.

LEMMA E 4: *Let*

$$\varepsilon \rightleftharpoons \exists Y((\pi(Y) = e) \wedge (\forall Y_1)((\pi(Y_1) = e) \to ((Y \circ Y_1 = Y_1) \vee (Y \circ Y_1 = Y)))).$$

Then $D(0,0) \models \varepsilon$ and $D(i_0, j_0) \models \neg\varepsilon$ for any $i_0 \in I$, $j_0 \in J$ such that $i_0 \neq 0$ or $j_0 \neq 0$.

Proof: For every $i_0 \in I$, $j_0 \in J$, $Y = (i,j)$, $Y_1 = (i_1, j_1)$ the equality $(Y \circ Y_1 = Y_1)$ means on $D(i_0, j_0)$ that $i = i_1$, i.e., that $\pi(Y) = q_{ji}$ and $\pi(Y_1) = q_{j_1 i_1}$ belong to the same i-th column of $P(i_0, j_0)$; the equality $(Y \circ Y_1 = Y)$ means on $D(i_0, j_0)$ that $j = j_1$, i.e., that $\pi(Y) = q_{ji}$ and $\pi(Y_1) = q_{j_1 i_1}$ belong to the same j-th row of $P(i_0, j_0)$. Therefore, informally speaking, the validity of ε on $D(i_0, j_0)$ means that in $P(i_0, j_0)$ there exist a row (number j) and a column (number i) such that the equality $q_{j_1 i_1} = e$ implies at least one of the equalities $i_1 = i$ or $j_1 = j$. That means, therefore, that every e of $P(i_0, j_0)$ belongs to the j-th row or to the i-th column.

The last condition is true for $i_0 = 0$, $j_0 = 0$, because in this case $P(i_0, j_0)$ is simply P, and we can take $i_1 \rightleftharpoons 0$, $j_1 \rightleftharpoons 0$. Therefore $D(0,0) \models \varepsilon$.

If $i_0 \neq 0$ or $j_0 \neq 0$, the condition considered above is false on $D(i_0, j_0)$. Indeed, in this case, in addition to the elements of the i_0-th column and to the elements of the j_0-th row of $P(i_0, j_0)$ (which all equal e because $P(i_0, j_0)$ is (i_0, j_0)-normalized), there exist $i_1 \neq i_0$ and $j_1 \neq j_0$ such that $q_{j_1 i_1} = e$ (see Lemma E 3). Therefore, if $i_0 \neq 0$ or $j_0 \neq 0$, then $D(i_0, j_0) \models \neg\varepsilon$. ∎

THEOREM 3: *$T_{<\cdot>}(C_0)$ is undecidable.*

Proof: Let φ be a closed formula from $L_{<\cdot,\circ,\pi>}$. Then from Lemma E 4 it follows that $D(0,0) \models \varphi$ iff $\Delta \models (\varepsilon \to \varphi)$. Therefore, the undecidability of $T_{<\cdot,\circ,\pi>}(D(0,0))$, established in Proposition E 1, implies the undecidability of $T_{<\cdot,\circ,\pi>}(\Delta)$. Hence, $T_{<\cdot>}(C_0)$ is undecidable by Theorem 2. ∎

References

[B] A. Baudisch, *Decidability and stability of free nilpotent Lee algebras and free nilpotent p-groups of finite exponent*, Annals of Mathematical Logic **23** (1982), 1–25.

[C-P] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups* **1**, American Mathematical Society, Providence, R.I., 1964.

[D] V. G. Durnev, *Systems of equations on free nilpotent groups*, in *Problems in Group Theory and Homological Algebra*, Yaroslav. Gos. Univ., Yaroslavl', 1981, pp. 66–69 (in Russian).

[J] P. R. Jones, *Completely simple semigroups: free products, free semigroups and varieties*, Proceedings of the Royal Society of Edinburgh. Section A **88** (1981), 293-314.

[Ma] A. I. Mal'cev, *Algebraic Systems*, Sprinter-Verlag, Berlin, 1973.

[Mash] G. I. Mashevitzky, *On bases of completely simple semigroups identities*, Semigroup Forum **30** (1984), 67–76.

[M] A. Mostowsky, *On direct products of theories*, Journal of Symbolic Logic **17** (1952), 1–31.

[M-R] A. G. Myasnikov and V. N. Remeslennikov, *Recursive p-adic numbers and elementary theories of finitely generated pro-p-groups*, Mathematics of the USSR-Izvestiya **30** (1988), 577–597.

[P-R] M. Petrich and N. R. Reilly, *Completely Regular Semigroups*, Canadian Mathematical Society, Series of Monographs and Advanced Texts **23**, Wiley-Interscience, New York, 1999.

[Ra] V. V. Rasin, *On the lattice of varieties of completely simple semigroups*, Semigroup Forum **17** (1979), 113–122.

[Ro] V. A. Roman'kov, *Equations in free metabelian groups*, Siberian Mathematical Journal **20** (1979), 469–471.

[R92] B. V. Rozenblat, *On the elementary theories of relatively free semigroups*, Abstracts of papers presented to the American Mathematical Society **13** (1992), 252.

[R93] B. V. Rozenblat, *On semigroup varieties in which every free semigroup has the solvable elementary theory*, Workshop on Semigroups and Connected Questions, Contributions, The Open University of Israel, Tel Aviv, 1993, pp. 25–28.

[Sv] L. N. Shevrin (ed.), *Sverdlovskaya Tetrad (Unsolved Problems of Semigroup Theory)*, 2nd edition, Sverdlovsk, 1979 (in Russian).